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2002 J. Phys. A: Math. Gen. 35 4949

(http://iopscience.iop.org/0305-4470/35/23/310)

J. Phys. A: Math. Gen. **35** (2002) 4949–4953 PII: S0305-4470(02)34319-1

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Received 25 February 2002, in final form 22 April 2002
Published 31 May 2002
Online at stacks.iop.org/JPhysA/35/4949

Abstract
It is shown that multi-dimensional generalization of the vector product is only possible in seven-dimensional space.

PACS number: 02.40.Dr

The three-dimensional vector product has proved to be useful in various physical problems. A natural question is whether multi-dimensional generalization of the vector product is possible. This apparently simple question has a somewhat unexpected answer, not widely known in the physics community, that generalization is only possible in seven-dimensional space. In mathematics this fact has been known since the 1940s [1], but only recently quite a simple proof (in comparison to previous ones) was given by Markus Rost [2]. Below I present a version of this proof to make it more accessible to physicists.

For contemporary physics a seven-dimensional vector product is not just of academic interest. It turned out that the corresponding construction is useful in considering self-dual Yang–Mills fields depending only upon time (Nahm equations) which, by themselves, originate in the context of M-theory [3, 4]. Other possible applications include Kaluza–Klein compactifications of $d = 11$, $N = 1$ supergravity [5]. That is why I think that this beautiful mathematical result should be known to a general audience of physicists.

Let us consider n-dimensional vector space $\mathbb{R}^n$ over the real numbers with the standard Euclidean scalar product. Which properties do we want the multi-dimensional bilinear vector product in $\mathbb{R}^n$ to satisfy? It is natural to choose as defining axioms the following (intuitively most evident) properties of the usual three-dimensional vector product:

$$\vec{A} \times \vec{A} = 0, \tag{1}$$

$$(\vec{A} \times \vec{B}) \cdot \vec{A} = (\vec{A} \times \vec{B}) \cdot \vec{B} = 0, \tag{2}$$

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|, \qquad \text{if } \vec{A} \cdot \vec{B} = 0. \tag{3}$$

Here $|\vec{A}|^2 = \vec{A} \cdot \vec{A}$ is the norm of the vector $\vec{A}$.

Then

$$0 = (\vec{A} + \vec{B}) \times (\vec{A} + \vec{B}) = \vec{A} \times \vec{B} + \vec{B} \times \vec{A}$$

shows that the vector product is anticommutative. By the same trick one can prove that $(\vec{A}\times\vec{B})\cdot\vec{C}$ is alternating in $\vec{A}$, $\vec{B}$, $\vec{C}$. For example

$$0=((\vec{A}+\vec{C})\times\vec{B})\cdot(\vec{A}+\vec{C})=(\vec{C}\times\vec{B})\cdot\vec{A}+(\vec{A}\times\vec{B})\cdot\vec{C}$$

shows that $(\vec{C}\times\vec{B})\cdot\vec{A}=-(\vec{A}\times\vec{B})\cdot\vec{C}$.

For any two vectors $\vec{A}$ and $\vec{B}$ the norm $|\vec{A}\times\vec{B}|^2$ is equal to

$$\left|\left(\vec{A}-\frac{\vec{A}\cdot\vec{B}}{|\vec{B}|^2}\vec{B}\right)\times\vec{B}\right|^2=\left|\vec{A}-\frac{\vec{A}\cdot\vec{B}}{|\vec{B}|^2}\,\vec{B}\right|^2|\vec{B}|^2=|\vec{A}|^2|\vec{B}|^2-(\vec{A}\cdot\vec{B})^2.$$

Therefore for any two vectors we should have

$$(\vec{A}\times\vec{B})\cdot(\vec{A}\times\vec{B})=(\vec{A}\cdot\vec{A})(\vec{B}\cdot\vec{B})-(\vec{A}\cdot\vec{B})^2. \tag{4}$$

Now consider

$$|\vec{A}\times(\vec{B}\times\vec{A})-(\vec{A}\cdot\vec{A})\vec{B}+(\vec{A}\cdot\vec{B})\vec{A}|^2$$
$$=|\vec{A}\times(\vec{B}\times\vec{A})|^2+|\vec{A}|^4|\vec{B}|^2-(\vec{A}\cdot\vec{B})^2|\vec{A}|^2-2|\vec{A}|^2(\vec{A}\times(\vec{B}\times\vec{A}))\cdot\vec{B}.$$

But this is zero because

$$|\vec{A}\times(\vec{B}\times\vec{A})|^2=|\vec{A}|^2|\vec{B}\times\vec{A}|^2=|\vec{A}|^4|\vec{B}|^2-(\vec{A}\cdot\vec{B})^2|\vec{A}|^2$$

and

$$(\vec{A}\times(\vec{B}\times\vec{A}))\cdot\vec{B}=(\vec{B}\times\vec{A})\cdot(\vec{B}\times\vec{A})=|\vec{A}|^2|\vec{B}|^2-(\vec{A}\cdot\vec{B})^2.$$

Therefore we have proved the identity

$$\vec{A}\times(\vec{B}\times\vec{A})=(\vec{A}\cdot\vec{A})\vec{B}-(\vec{A}\cdot\vec{B})\vec{A}. \tag{5}$$

Note that the arrangement of the brackets on the lhs is, in fact, irrelevant because the vector product is anticommutative.

However, the familiar identity

$$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B}) \tag{6}$$

does not follow in general from the intuitively evident properties (1)–(3) of the vector product [2]. To show this, let us introduce the ternary product [6] (which is zero if (6) is valid)

$$\{\vec{A},\vec{B},\vec{C}\}=\vec{A}\times(\vec{B}\times\vec{C})-\vec{B}(\vec{A}\cdot\vec{C})+\vec{C}(\vec{A}\cdot\vec{B}).$$

Equation (5) implies that this ternary product is alternating in its arguments. For example

$$0=\{\vec{A}+\vec{B},\vec{A}+\vec{B},\vec{C}\}=\{\vec{A},\vec{B},\vec{C}\}+\{\vec{B},\vec{A},\vec{C}\}.$$

If $\vec{e}_i$, $i=1\div n$ is an orthonormal basis in the vector space, then

$$(\vec{e}_i\times\vec{A})\cdot(\vec{e}_i\times\vec{B})=((\vec{e}_i\times\vec{B})\times\vec{e}_i)\cdot\vec{A}=[\vec{B}-(\vec{B}\cdot\vec{e}_i)\vec{e}_i]\cdot\vec{A}$$

and, therefore,

$$\sum_{i=1}^{n}(\vec{e}_i\times\vec{A})\cdot(\vec{e}_i\times\vec{B})=(n-1)\vec{A}\cdot\vec{B}. \tag{7}$$

Using this identity we obtain

$$\sum_{i=1}^{n}\{\vec{e}_i,\vec{A},\vec{B}\}\cdot\{\vec{e}_i,\vec{C},\vec{D}\}=(n-5)(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})$$
$$+2(\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})-2(\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}). \tag{8}$$

Hence

$$\sum_{i,j=1}^{n} \{\vec{e}_i, \vec{e}_j, \vec{A}\} \cdot \{\vec{e}_i, \vec{e}_j, \vec{B}\} = (n-1)(n-3)\vec{A} \cdot \vec{B} \tag{9}$$

and [6]

$$\sum_{i,j,k=1}^{n} \{\vec{e}_i, \vec{e}_j, \vec{e}_k\} \cdot \{\vec{e}_i, \vec{e}_j, \vec{e}_k\} = n(n-1)(n-3). \tag{10}$$

The last equation shows that there exists some $\{\vec{e}_i, \vec{e}_j, \vec{e}_k\}$ that is not zero if $n > 3$. So equation (6) is valid only for the usual three-dimensional vector product (the $n = 1$ case is, of course, not interesting because it corresponds to a identically vanishing vector product). Surprisingly, we do not have much choice for n even in the case when the validity of (6) is not required. In fact, the space dimension n should satisfy [2] (see also [7])

$$n(n-1)(n-3)(n-7) = 0. \tag{11}$$

To prove this statement, let us note that using

$$\begin{aligned}\vec{A} \times (\vec{B} \times \vec{C}) + (\vec{A} \times \vec{B}) \times \vec{C} &= (\vec{A} + \vec{C}) \times \vec{B} \times (\vec{A} + \vec{C}) - \vec{A} \times \vec{B} \times \vec{A} - \vec{C} \times \vec{B} \times \vec{C} \\ &= 2\vec{A} \cdot \vec{C}\vec{B} - \vec{A} \cdot \vec{B}\,\vec{C} - \vec{B} \cdot \vec{C}\vec{A}\end{aligned}$$

and

$$\begin{aligned}\vec{A} \times (\vec{B} \times (\vec{C} \times \vec{D})) = \tfrac{1}{2}[&\vec{A} \times (\vec{B} \times (\vec{C} \times \vec{D})) + (\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) \\ &- (\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) - ((\vec{A} \times \vec{B}) \times \vec{C}) \times \vec{D} + ((\vec{A} \times \vec{B}) \times \vec{C}) \times \vec{D} \\ &+ (\vec{A} \times (\vec{B} \times \vec{C})) \times \vec{D} - (\vec{A} \times (\vec{B} \times \vec{C})) \times \vec{D} - \vec{A} \times ((\vec{B} \times \vec{C}) \times \vec{D}) \\ &+ \vec{A} \times ((\vec{B} \times \vec{C}) \times \vec{D}) + \vec{A} \times (\vec{B} \times (\vec{C} \times \vec{D}))]\end{aligned}$$

we can check the equation

$$\begin{aligned}\vec{A} \times \{\vec{B}, \vec{C}, \vec{D}\} &= -\{\vec{A}, \vec{B}, \vec{C} \times \vec{D}\} + \vec{A} \times (\vec{B} \times (\vec{C} \times \vec{D})) - \{\vec{A}, \vec{C}, \vec{D} \times \vec{B}\} \\ &\quad + \vec{A} \times (\vec{C} \times (\vec{D} \times \vec{B})) - \{\vec{A}, \vec{D}, \vec{B} \times \vec{C}\} + \vec{A} \times (\vec{D} \times (\vec{B} \times \vec{C})) \\ &= -\{\vec{A}, \vec{B}, \vec{C} \times \vec{D}\} - \{\vec{A}, \vec{C}, \vec{D} \times \vec{B}\} - \{\vec{A}, \vec{D}, \vec{B} \times \vec{C}\} + 3\vec{A} \times \{\vec{B}, \vec{C}, \vec{D}\}.\end{aligned}$$

The last step follows from

$$\begin{aligned}3\{\vec{B}, \vec{C}, \vec{D}\} &= \{\vec{B}, \vec{C}, \vec{D}\} + \{\vec{C}, \vec{D}, \vec{B}\} + \{\vec{D}, \vec{B}, \vec{C}\} \\ &= \vec{B} \times (\vec{C} \times \vec{D}) + \vec{C} \times (\vec{D} \times \vec{B}) + \vec{D} \times (\vec{B} \times \vec{C}).\end{aligned}$$

Therefore the ternary product satisfies an interesting identity

$$2\vec{A} \times \{\vec{B}, \vec{C}, \vec{D}\} = \{\vec{A}, \vec{B}, \vec{C} \times \vec{D}\} + \{\vec{A}, \vec{C}, \vec{D} \times \vec{B}\} + \{\vec{A}, \vec{D}, \vec{B} \times \vec{C}\}. \tag{12}$$

Hence we should have

$$4 \sum_{i,j,k,l=1}^{n} |\vec{e}_i \times \{\vec{e}_j, \vec{e}_k, \vec{e}_l\}|^2 = \sum_{i,j,k,l=1}^{n} |\{\vec{e}_i, \vec{e}_j, \vec{e}_k \times \vec{e}_l\} + \{\vec{e}_i, \vec{e}_k, \vec{e}_l \times \vec{e}_j\} + \{\vec{e}_i, \vec{e}_l, \vec{e}_j \times \vec{e}_k\}|^2.$$

The lhs is easily calculated by means of (7) and (10):

$$4 \sum_{i,j,k,l=1}^{n} |\vec{e}_i \times \{\vec{e}_j, \vec{e}_k, \vec{e}_l\}|^2 = 4n(n-1)^2(n-3).$$

To calculate the rhs the following identity is useful:

$$\sum_{i,j=1}^{n} \{\vec{e}_i, \vec{e}_j, \vec{A}\} \cdot \{\vec{e}_i, \vec{e}_j \times \vec{B}, \vec{C}\} = -(n-3)(n-6)\vec{A} \cdot (\vec{B} \times \vec{C}) \tag{13}$$

which follows from (8) and from the identity

$$\sum_{i=1}^{n}(\vec{e}_i \times \vec{A}) \cdot ((\vec{e}_i \times \vec{B}) \times \vec{C})$$

$$= \sum_{i=1}^{n}(\vec{e}_i \times \vec{A}) \cdot [2\vec{e}_i \cdot \vec{C}\vec{B} - \vec{B} \cdot \vec{C}\vec{e}_i - \vec{e}_i \cdot \vec{B}\vec{C} - \vec{e}_i \times (\vec{B} \times \vec{C})]$$

$$= -(n-4)\vec{A} \cdot (\vec{B} \times \vec{C}).$$

Now, with (9) and (13) at hand, it becomes an easy task to calculate

$$\sum_{i,j,k,l=1}^{n} |\{\vec{e}_i, \vec{e}_j, \vec{e}_k \times \vec{e}_l\} + \{\vec{e}_i, \vec{e}_k, \vec{e}_l \times \vec{e}_j\} + \{\vec{e}_i, \vec{e}_l, \vec{e}_j \times \vec{e}_k\}|^2$$

$$= 3n(n-1)^2(n-3) + 6n(n-1)(n-3)(n-6) = 3n(n-1)(n-3)(3n-13).$$

Therefore we should have

$$4n(n-1)^2(n-3) = 3n(n-1)(n-3)(3n-13).$$

But

$$3n(n-1)(n-3)(3n-13) - 4n(n-1)^2(n-3) = 5n(n-1)(n-3)(n-7)$$

and hence (11) follows.

As we see, the space dimension must be equal to the magic number seven if unique generalization of the ordinary three-dimensional vector product is possible.

So far we only have shown that a seven-dimensional vector product can exist in principle. What about its detailed realization? To answer this question, it is useful to realize that the vector products are closely related to composition algebras [1] (in fact, these two notions are equivalent [2]). Namely, for any composition algebra with unit element e we can define the vector product in the subspace orthogonal to e by $x \times y = \frac{1}{2}(xy - yx)$. Therefore, from a viewpoint of composition algebra, the vector product is just the commutator divided by two. According to Hurwitz theorem [8] the only composition algebras are real numbers, complex numbers, quaternions and octonions. The first two of them give identically zero vector products. Quaternions produce the usual three-dimensional vector product. The seven-dimensional vector product is generated by octonions [9]. It is interesting to note that this seven-dimensional vector product is covariant with respect to the smallest exceptional Lie group G_2 [10] which is the automorphism group of octonions.

Using the octonion multiplication table [9] one can realize the seven-dimensional vector product as follows:

$$\vec{e}_i \times \vec{e}_j = \sum_{k=1}^{7} f_{ijk}\vec{e}_k, \qquad i, j = 1, 2, \ldots, 7, \tag{14}$$

where f_{ijk} is a totally antisymmetric G_2-invariant tensor and the only nonzero independent components are

$$f_{123} = f_{246} = f_{435} = f_{651} = f_{572} = f_{714} = f_{367} = 1.$$

Note that, in contrast to the three-dimensional case, $f_{ijk}f_{kmn} \neq \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$. Instead we have

$$f_{ijk}f_{kmn} = g_{ijmn} + \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \tag{15}$$

where

$$g_{ijmn} = \vec{e}_i \cdot \{\vec{e}_j, \vec{e}_m, \vec{e}_n\}.$$

In fact, g_{ijmn} is a totally antisymmetric G_2-invariant tensor. For example

$$\begin{aligned} g_{ijmn} &= \vec{e}_i \cdot \{\vec{e}_j, \vec{e}_m, \vec{e}_n\} = -\vec{e}_i \cdot \{\vec{e}_m, \vec{e}_j, \vec{e}_n\} \\ &= -\vec{e}_i \cdot (\vec{e}_m \times (\vec{e}_j \times \vec{e}_n)) + (\vec{e}_i \cdot \vec{e}_j)(\vec{e}_m \cdot \vec{e}_n) - (\vec{e}_i \cdot \vec{e}_n)(\vec{e}_m \cdot \vec{e}_j) \\ &= -\vec{e}_m \cdot (\vec{e}_i \times (\vec{e}_n \times \vec{e}_j)) + (\vec{e}_i \cdot \vec{e}_j)(\vec{e}_m \cdot \vec{e}_n) - (\vec{e}_i \cdot \vec{e}_n)(\vec{e}_m \cdot \vec{e}_j) \\ &= -\vec{e}_m \cdot \{\vec{e}_i, \vec{e}_n, \vec{e}_j\} = -\vec{e}_m \cdot \{\vec{e}_j, \vec{e}_i, \vec{e}_n\} = -g_{mjin}. \end{aligned}$$

The only nonzero independent components are

$$g_{1254} = g_{1267} = g_{1364} = g_{1375} = g_{2347} = g_{2365} = g_{4576} = 1.$$

In conclusion, generalization of the vector product we have considered is only possible in seven-dimensional space and is closely related to octonions—the largest composition algebra which ties together many exceptional structures in mathematics [10]. In a general case of p-fold vector products other options arise [1, 5, 11]. We recommend that the interested reader consults the references to explore these possibilities and their possible physical applications.

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